Wolf – Man in Pool Problem
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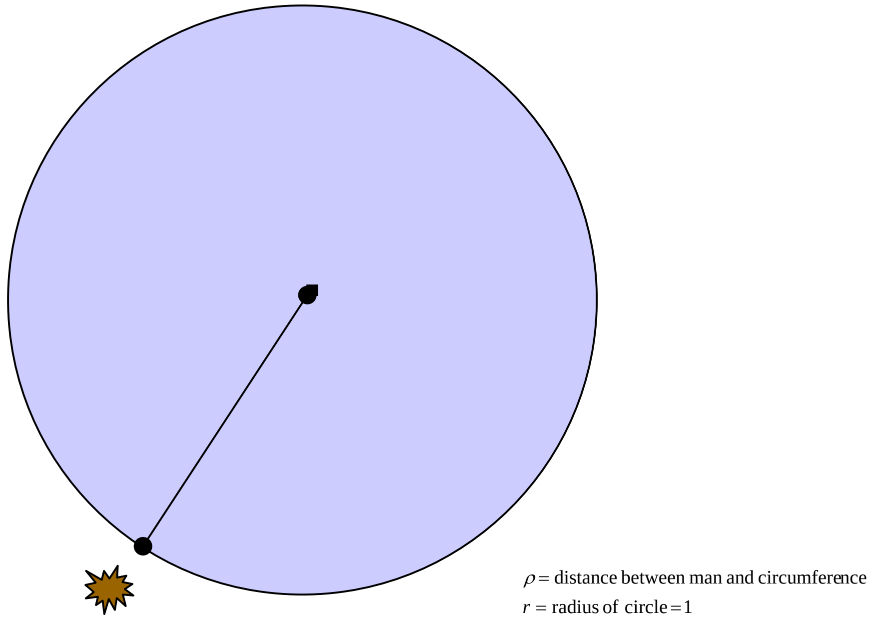


I. Setup

A man, located in the center of a circular pool (Circle-A) is being chased by a wolf, who is running along the circumference of the pool.

The man can swim at a speed that is $\frac{1}{4}$ the velocity of the wolf.

How does the man escape?

II. Proposed Solution

let ρ = distance between man and the edge of the pool
let r = radius of the circle, which we will assume to be the unit circle, $r = 1$

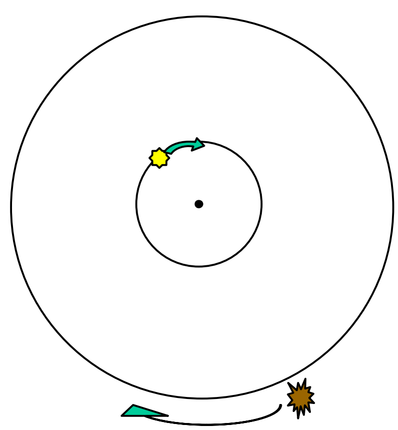
Question: At what radius does the man have the ability to exactly mirror the θ position of the wolf?

$$v_w = \frac{d_w}{t_w} \;\&\; v_m = \frac{1}{4}v_w = \frac{d_w}{4t_w} \equiv \frac{d}{4w}$$

The velocity of the wolf (distance/time) is the circumference divided by the time it will take the wolf to travel, which we will normalize as 1. We want to find some ρ s.t. the time it takes the man to circumnavigate the circle defined by radius $(1-\rho)$ = time it takes the wolf to travel around the larger circle. Lets use $\overline{\rho}$ to denote that distance. Let's further denote the circle created by this radius to be circle-B.

$$\frac{2\overline{\rho}\pi}{t} = \frac{1}{4}\left(\frac{2r\pi}{t}\right)$$

$$\Rightarrow \overline{\rho} = \frac{1}{4}r = \frac{1}{4}$$



So, at a distance from the center of 0.25, the man can maintain a position exactly opposite the wolf, no matter what the wolf does.

So, if I can get to the rho-compliment radius (1/4) while keeping the wolf on the other 'side' of the pond, then I can always maintain that position.

LEMMA 1 – I can get to such a position.

PROPOSITION I - From that position, I can always escape.

PROOF P-1:

(1) If we normalize the man's speed, then he, if he heads to the tangent point on the edge of the lake, will be a distance *d* units, which is ρ =0.75.

(2) The wolf must travel exactly half of the circumference to catch the man:

$$\begin{aligned} d_w &= \frac{1}{2} circumference \\ &= \frac{1}{2}(\pi 2r) \\ &= r\pi \\ &= \pi \end{aligned}$$

(3) The time that the wolf will take to travel a distance π will be normalized.

$$t_{wolf} = 1$$

(4) The man travels one-forth of the distance of the wolf in the same unit-time:

$$\frac{d_{man}}{t_{wolf}} = \frac{1}{4}\frac{d_{wolf}}{t_{wolf}}$$

normalized,

$$d_{man} = \frac{1}{4}d_{wolf}$$

(5) Plug in $d_{wolf} = \pi$ and you see that in the same unit time, a man would travel:

$$\begin{aligned} d_{man} &= \frac{1}{4}\pi \\ &\approx 0.7854 \end{aligned}$$

(6) BUT, note that the man only has to travel the distance 0.75 to escape, therefore he can escape.

//QED-P1

Ok, so let's go back to the lemma to show that in fact we can get the man into that special position of "opposite" the wolf on the Circle-B.

PROOF L-1:

(1) Circle-B is the max effective radius that a swimmer can swim s.t. he will be able to sweep out the same θ as the wolf running on the outside Circle-A.

(2) Any sub-circle with same origin, and strictly smaller radius, of Circle-B, *must* have a smaller circumference. Therefore, the man would be able to cover a larger θ than the wolf in the same unit-time.

(3) (informal, but obvious statement) The man can reach Circle-B simply by going to the perimeter of Circle B in a spiral like pattern, always maintaining the relationship:

$$\theta_{man} = \theta_{wolf} + \pi$$

(4) If you are bored, I am sure you could write a parametric representation of $\theta(t)$ to get the exact spiral for the man to travel at.

Once he is on the perimeter of Circle-B, where $\theta_{man} = \theta_{wolf} + \pi$, then PROP-1 shows that the man can escape.

III. Nash Equilibrium Solution

For this solution to be optimal, we must make sure that the wolf is optimizing given the player's play, and that the player is optimizing on the wolf's play.

So, it is clear to see that if the wolf does not attempt to chase the player, the player will, in fact, have an easier time reaching the boundary perimeter (from which he can escape given any play from the wolf).

So, the wolf's optimal play is to chase the player around his/her spiral. However, given that play, the player can always reach the boundary (Lemma 1).

IV – Other Solutions

It is also interesting to note the following proof which allows the player to win, *regardless of the ability to reach the perimeter*.

The perimeter of Circle-B is the distance, rho, s.t. the player can circumnavigate the pool at the same rate as the wolf. As previously stated, any distance less than rho, the player can "sweep" out a larger theta-angle than the wolf (effectively, he can "run faster"). Therefore, the player can always reach his optimal position ($\theta + \pi$) because he can "outrun" the wolf.

Note further, that the proof that the player can reach the perimeter of the circle has a "give" in it… that means that at $\rho - \varepsilon$ distance from the center, in fact the player can reach the edge of the pool.

Q.E.D.